

# VERTEX OPERATOR REPRESENTATIONS OF TYPE $C_l^{(1)}$ AND PRODUCT-SUM IDENTITIES

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**ABSTRACT.** The purposes of this work are to construct a class of homogeneous vertex representations of  $C_l^{(1)}$  ( $l \geq 2$ ), and to derive a series of product-sum identities. These identities have fine interpretation in number theory.

*Key words:* Vertex operator, representations, product-sum identities.

## 1. INTRODUCTION

It is well known that there is a close relationship between representations of affine Lie algebras and combinatorics. For example, the Jacobi triple product identity can be obtained as the Weyl-Kac denominator formula for the affine Lie algebra  $\widehat{sl}_2$  ([7]). The famous Rogers-Ramanujan identities can be realized from the character formula of certain level three representations [8]. Like the Jacobi triple product identity, the quintuple product identity is also equivalent to the Weyl-Kac denominator formula for the affine Lie algebra  $A_2^{(2)}$ . In [6], the following infinite product

$$(1.1) \quad \prod_{n=1}^{\infty} \frac{1}{(1 - q^{6n-1})(1 - q^{6n-5})}$$

is expressed by a sum of two other infinite products in four different ways.

I. Schur [12] (see also [1]) was probably the first person who studied the partitions described by (1.1). He showed that the number of partitions of  $n$  into parts congruent to  $\pm 1 \pmod{6}$  is equal to the number of partitions of  $n$  into distinct parts congruent to  $\pm 1 \pmod{3}$  and is also equal to the number of partitions of  $n$  into parts that differ at least 3 with added condition that difference between multiples of 3 is at least 6. His first result can be briefly described by

$$(1.2) \quad \prod_{n=1}^{\infty} \frac{1}{(1 - q^{6n-1})(1 - q^{6n-5})} = \prod_{n=1}^{\infty} \frac{(1 + q^n)}{(1 + q^{3n})}.$$

Motivated by product-sum identity provided by [6], we study a generalized product-sum relations of some special partitions. Our method uses the vertex representations of affine Lie algebras of type  $C_l^{(1)}$ . For the related topics, one can refer [4], [5], [10], [11], [13] and references therein.

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**Theorem 1.1.** *For any odd  $l \geq 3$ , the following product-sum identity holds:*

$$\prod_{n=1}^{\infty} \frac{(1+q^n)}{(1+q^{ln})} = \sum_{s=0}^{\frac{l-1}{2}} q^{\frac{(l-2s)^2-1}{8}} \prod_{n \not\equiv \pm(s+1), 0 \pmod{l+2}} \frac{1}{(1-q^{2n})(1-q^{ln})},$$

particularly, it covers the first result of [6] when  $l = 3$ .

Our result in Theorem 1.1 implies the following partition theorem:

**Theorem 1.2.** *Suppose that  $l = 2r + 1 \geq 3$  is an odd number,  $A_l(n)$  is the number of partitions of  $n$  into distinct parts without multiples of  $l$ , and  $B_{l,s}(n)$  is the number of partitions of  $n$  into*

$$2k_1 + \cdots + 2k_i + lr_1 + \cdots + lr_j + \frac{(l-2s)^2-1}{8}$$

with constraints  $k_p, r_p \not\equiv \pm(s+1), 0 \pmod{l+2}$ . Then for any positive integer  $n$ , we have

$$A_l(n) = B_{l,0}(n) + B_{l,1}(n) + \cdots + B_{l,r}(n).$$

*Proof.* Let  $1 + \sum_{n=1}^{\infty} A_n a^n$  be the power series of  $\prod_{n=1}^{\infty} \frac{(1+q^n)}{(1+q^{ln})}$ . Because

$$\begin{aligned} \prod_{n=1}^{\infty} \frac{(1+q^n)}{(1+q^{ln})} &= \prod_{n \geq 1 \text{ is not a multiple of } l} (1+q^n) \\ &= \sum_{\substack{n_1 > n_2 > \cdots > n_k \geq 1 \\ n_i \text{ is not a multiple of } l, k \geq 0}} q^{n_1 + \cdots + n_k}, \end{aligned}$$

Then  $A_n$  is the number of partitions of  $n$  into distinct parts without multiples of  $l$  and  $A_n = A(n)$ .

A similar argument on  $B_{l,s}(n)$  shows that Theorem 1.1 is equivalent to the relation

$$A_l(n) = B_{l,0}(n) + B_{l,1}(n) + \cdots + B_{l,r}(n), \quad \text{for all positive integer } n.$$

□

For example,

$A_5(15) = 16$	$B_{5,0}(15) = 3$	$B_{5,1}(15) = 7$	$B_{5,2}(15) = 6$
$1 + 14$	$1 + 6 + 8$	$2(2 \times 3) + 3$	$2(1 \times 7) + 1$
$2 + 13$	$2 + 4 + 9$	$2(2 + 4) + 3$	$2(1 \times 4 + 3) + 1$
$3 + 12$	$2 + 6 + 7$	$2(3 + 3) + 3$	$2(1 \times 3 + 2) + 1$
$4 + 11$	$3 + 4 + 8$		$2(1 + 3 \times 2) + 1$
$6 + 9$	$1 + 2 + 3 + 9$		$2(5) + 5(1)$
$7 + 8$	$1 + 2 + 4 + 8$		$2(1 \times 3 + 4) + 1$
$1 + 2 + 12$	$1 + 3 + 4 + 7$		$2(1 + 6) + 1$
$1 + 3 + 11$	$2 + 3 + 4 + 6$		$2(3 + 4) + 1$
			$5(1 + 2)$
			$2(1 \times 2) + 5(1 \times 2)$

Table 1 lists the values of  $A_5(n), B_{5,0}(n), B_{5,1}(n), B_{5,2}(n)$  for  $n \leq 15$ .

$n$	$A_5(n)$	$B_{5,0}(n)$	$B_{5,1}(n)$	$B_{5,2}(n)$
1	1	0	1	0
2	1	0	0	1
3	2	1	1	0
4	2	0	0	2
5	2	0	1	1
6	3	0	1	2
7	4	1	2	1
8	4	0	1	3
9	6	1	3	2
10	7	0	1	6
11	8	2	4	2
12	10	0	2	8
13	12	3	6	3
14	14	0	3	11
15	16	3	7	6

**Table 1**

The above results will be proved by the irreducible decompositions of vertex module  $V(P) = S(\widehat{H}^-) \otimes \mathbb{C}[P]$  of  $C_l^{(1)}$ , where  $1 \otimes 1$  has weight  $\Lambda_0$ . If we assume that  $1 \otimes 1$  has weight  $\Lambda_1$ , then our method also gives the following result:

**Theorem 1.3.** *For any even  $l \geq 2$ , the following product-sum identity holds:*

$$\prod_{n=1}^{\infty} \frac{(1 + q^{n-\frac{1}{2}})^2}{(1 + q^n)(1 + q^{ln})} = \frac{\prod_{n \geq 1} (1 - q^{l(\frac{l+2}{2})(2n-1)})(1 - q^{(l+2)(2n-1)})}{\prod_{(\frac{l}{2}+1) \nmid n} (1 - q^{2n})(1 - q^{ln})} \\ + 2 \sum_{s=0}^{\frac{l}{2}-1} q^{\frac{(l-2s)^2}{8}} \prod_{n \not\equiv \pm(s+1), 0 \pmod{l+2}} \frac{1}{(1 - q^{2n})(1 - q^{ln})},$$

or equivalently,

$$\prod_{n=1}^{\infty} \frac{(1 + q^{2n-1})^2}{(1 + q^{2n})(1 + q^{2ln})} = \frac{\prod_{n \geq 1} (1 - q^{l(l+2)(2n-1)})(1 - q^{2(l+2)(2n-1)})}{\prod_{(\frac{l}{2}+1) \nmid n} (1 - q^{4n})(1 - q^{2ln})} \\ + 2 \sum_{s=0}^{\frac{l}{2}-1} q^{\frac{(l-2s)^2}{4}} \prod_{n \not\equiv \pm(s+1), 0 \pmod{l+2}} \frac{1}{(1 - q^{4n})(1 - q^{2ln})}.$$

Throughout the paper, we let  $\mathbb{C}, \mathbb{Z}$  present the set of complex numbers and the set of integers, respectively.

## 2. AFFINE LIE ALGEBRA OF TYPE $C_l^{(1)}$

**2.1.** Let  $\dot{\mathcal{G}}$  be a finite-dimensional simple Lie algebra of type  $C_l$ ,  $A = \mathbb{C}[t^{\pm 1}]$  the ring of Laurent polynomials in variable  $t$ . Then the affine Lie algebra of type  $C_l^{(1)}$  is the vector space

$$\tilde{\mathcal{G}} = \dot{\mathcal{G}} \otimes A \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

with Lie bracket:

$$(2.1) \quad [x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m(x|y)\delta_{m+n,0}c,$$

$$(2.2) \quad [c, \mathcal{G}] = 0,$$

$$(2.3) \quad [d, x \otimes t^m] = mx \otimes t^m,$$

where  $x, y \in \dot{\mathcal{G}}$ ,  $m, n \in \mathbb{Z}$  and  $(\cdot|\cdot)$  is a nondegenerate invariant normalized symmetric bilinear form on  $\dot{\mathcal{G}}$ .

**2.2.** Suppose that  $\dot{H}$  is a Cartan subalgebra of  $\dot{\mathcal{G}}$ , and  $\dot{H}^*$  the dual space of  $\dot{H}$ . Then there exists an inner product  $(\cdot|\cdot)|_{\dot{H}_{\mathbf{R}}^*}$  and an orthogonal normal basis  $\{e_1, e_2, \dots, e_l\}$  in Euclidian space  $\dot{H}_{\mathbf{R}}^*$  such that the simple root system

$$\Pi = \left\{ \alpha_1 = \frac{1}{\sqrt{2}}(e_1 - e_2), \dots, \alpha_{l-1} = \frac{1}{\sqrt{2}}(e_{l-1} - e_l), \alpha_l = \sqrt{2}e_l \right\},$$

the short root system

$$\dot{\Delta}_S = \left\{ \pm \frac{1}{\sqrt{2}}(e_i - e_j), \pm \frac{1}{\sqrt{2}}(e_i + e_j) \mid 1 \leq i < j \leq l \right\},$$

where  $\frac{1}{\sqrt{2}}(e_i - e_j) = \alpha_i + \dots + \alpha_{j-1}$  for  $1 \leq i < j \leq l$ ,  $\frac{1}{\sqrt{2}}(e_i + e_l) = \alpha_i + \dots + \alpha_l$  for  $1 \leq i < l$ ,  $\frac{1}{\sqrt{2}}(e_i + e_j) = \alpha_i + \dots + \alpha_{j-1} + 2\alpha_j + \dots + 2\alpha_{l-1} + \alpha_l$  for  $1 \leq i < j < l$ ;

and the long root system

$$\dot{\Delta}_L = \left\{ \pm \sqrt{2}e_i \mid 1 \leq i \leq l \right\},$$

where  $\sqrt{2}e_i = 2\alpha_i + \dots + 2\alpha_{l-1} + \alpha_l$  for  $1 \leq i < l$ .

Then the root lattice is

$$Q = \bigoplus_{i=1}^l \mathbb{Z}\alpha_i$$

and  $(\alpha_i|\alpha_i) = 1$  ( $1 \leq i \leq l-1$ ), and  $(\alpha_l|\alpha_l) = 2$ .

Let  $\gamma : \dot{H} \longrightarrow \dot{H}^*$  be the linear isomorphism such that

$$\alpha_i(\gamma^{-1}(\alpha_j)) = (\alpha_i|\alpha_j), \quad i, j = 1, \dots, l,$$

and

$$\gamma(\alpha_i^\vee) = 2\alpha_i \quad (i = 1, \dots, l-1), \quad \gamma(\alpha_l^\vee) = \alpha_l.$$

Then we have  $(\alpha_i^\vee|\alpha_j^\vee) = (\gamma(\alpha_i^\vee)|\gamma(\alpha_j^\vee))$ . As usual, we identify  $\dot{H}$  with  $\dot{H}^*$  via  $\gamma$ , i.e.,  $\alpha^\vee = \frac{2\alpha}{(\alpha|\alpha)}$ .

For any weight  $\Lambda \in (\dot{H} \oplus \mathbb{C}c \oplus \mathbb{C}d)^*$ , let  $L(\Lambda)$  denote the irreducible highest weight  $\tilde{\mathcal{G}}$ -module with highest weight  $\Lambda$ .

**2.3.** Define a 2-cocycle  $\epsilon_0 : Q \times Q \longrightarrow \{\pm 1\}$  by

$$\epsilon_0(a+b, c) = \epsilon_0(a, c)\epsilon_0(b, c), \quad \epsilon_0(a, b+c) = \epsilon_0(a, b)\epsilon_0(a, c), \quad a, b, c \in L,$$

and

$$\epsilon_0(\alpha_i, \alpha_j) = \begin{cases} -1, & i = j+1, \\ 1, & \text{other pairs } (i, j), \end{cases}$$

Let  $P = \bigoplus_{i=1}^l \mathbb{Z}\alpha_i \oplus \frac{1}{2}\mathbb{Z}\alpha_l$ . Extend  $\epsilon_0$  to  $Q \times P$  with

$$\epsilon_0(\alpha_i, \frac{1}{2}\alpha_l) = 1.$$

**2.4.** For  $\alpha = \sum_{i=1}^l k_i \alpha_i \in \dot{\Delta} \cup \{0\}$ , define maps  $p : \dot{\Delta} \cup \{0\} \rightarrow \dot{\Delta}_S \cup \{0\}$  and  $s : \dot{\Delta} \cup \{0\} \rightarrow \dot{Q}_L$  by:

$$p \left( \sum_{i=1}^l k_i \alpha_i \right) = \sum_{i=1}^{l-1} \rho(k_i) \alpha_i, \quad s \left( \sum_{i=1}^l k_i \alpha_i \right) = \sum_{i=1}^{l-1} (k_i - \rho(k_i)) \alpha_i,$$

where  $\dot{Q}_L = \text{Span}_{\mathbb{Z}} \dot{\Delta}_L$  and  $\rho(k_i) \in \{0, 1\}$  such that  $\rho(k_i) \equiv k_i \pmod{2}$ . It is straightforward to check the following statements.

**Lemma 2.1.** (i)  $p(\dot{\Delta}_L \cup \{0\}) = 0$ , and  $p(-\alpha) = p(\alpha)$  for any  $\alpha \in \dot{\Delta}_S$ .

(ii) Suppose that  $\alpha, \beta, \alpha + \beta \in \dot{\Delta}$ , then we have:

(1) if  $\alpha \in \dot{\Delta}_L$ , then  $(\alpha | \beta) = -1$ ,  $p(\alpha + \beta) = p(\beta)$ ,  $s(\alpha + \beta) = s(\alpha) + s(\beta)$ ;

(2) if  $\alpha, \beta \in \dot{\Delta}_S$ ,  $\alpha + \beta \in \dot{\Delta}_L$ , then  $(\alpha | \beta) = 0$ ,  $p(\alpha) = p(\beta)$ ,  $s(\alpha + \beta) - s(\alpha) - s(\beta) = 2p(\alpha)$ ;

(3) if  $\alpha, \beta, \alpha + \beta \in \dot{\Delta}_S$ , then  $(\alpha | \beta) = -\frac{1}{2}$  and  $|(p(\alpha) | p(\beta))| = \frac{1}{2}$ ; Moreover,

(a)  $(p(\alpha) | p(\beta)) = \frac{1}{2}$ , then  $p(\alpha + \beta) = p(\alpha) - p(\beta)$ ,  $s(\alpha + \beta) - s(\alpha) - s(\beta) = 2p(\beta)$ , or  $p(\alpha + \beta) = -p(\alpha) + p(\beta)$ ,  $s(\alpha + \beta) - s(\alpha) - s(\beta) = 2p(\alpha)$ ;

(b) if  $(p(\alpha) | p(\beta)) = -\frac{1}{2}$ , then  $p(\alpha + \beta) = p(\alpha) + p(\beta)$ ,  $s(\alpha + \beta) = s(\alpha) + s(\beta)$ .

(iii) For any  $\alpha \in \dot{\Delta}$ , we have:

(1)  $s(\alpha) \in \{\pm\sqrt{2}(e_i - e_l) \mid 1 \leq i \leq l\} \subset \dot{Q}_L$ ;

(2)  $p(\alpha) \in \{\frac{1}{\sqrt{2}}(e_i - e_j) \mid 1 \leq i \leq j \leq l\} \subset \dot{\Delta}_S \cup \{0\}$ ;

(3)  $s(\alpha) + s(-\alpha) = -2p(\alpha) \in \dot{Q}_L$ ;

(4)  $\alpha \pm p(\alpha) \in \dot{Q}_L$ .

**2.5.** Define a map  $f : Q \times Q \rightarrow \{\pm 1\}$  by

$$f(\alpha, \beta) = (-1)^{(s(\alpha)|\beta) + (p(\alpha)|p(\beta) + p(\alpha + \beta))}.$$

Set  $\epsilon = \epsilon_0 \circ f$ , then  $\epsilon : Q \times Q \rightarrow \{\pm 1\}$  is still a 2-cocycle, which has the property (ii) in the following Lemma.

**Lemma 2.2.** (i) For  $\alpha, \beta \in \dot{\Delta}$ , we have

$$\epsilon_0(\alpha, \beta) = (-1)^{(\alpha|\beta) + (p(\alpha)|p(\beta)) + (s(\alpha)|\beta) + (s(\beta)|\alpha)} \cdot \epsilon_0(\beta, \alpha).$$

(ii) For  $\alpha, \beta, \alpha + \beta \in \dot{\Delta}$ , we have  $\epsilon(\alpha, \beta) = -\epsilon(\beta, \alpha)$ .

**2.6.** We have

**Proposition 2.3.** The affine Lie algebra  $\tilde{\mathcal{G}}$  of type  $C_l^{(1)}$  has a system of generators

$$\{\alpha_i^\vee \otimes t^n, e_\alpha \otimes t^n \mid 1 \leq i \leq l, n \in \mathbb{Z}\}$$

and  $c, d$  with relations

$$\begin{aligned} [\alpha_i^\vee \otimes t^m, \alpha_j^\vee \otimes t^n] &= m(\alpha_i^\vee | \alpha_j^\vee) \delta_{m+n, 0} c, \\ [\alpha_i^\vee \otimes t^m, e_\alpha \otimes t^n] &= \alpha(\alpha_i^\vee) e_\alpha \otimes t^{m+n}, \\ [e_\alpha \otimes t^m, e_{-\alpha} \otimes t^n] &= \epsilon(\alpha, -\alpha) \frac{2}{(\alpha | \alpha)} [\gamma^{-1}(\alpha) \otimes t^{m+n} + m \delta_{m+n, 0} c], \\ [e_\alpha \otimes t^m, e_\beta \otimes t^n] &= \epsilon(\alpha, \beta) (1 + \delta_{1, (p(\alpha) | p(\beta))}) e_{\alpha+\beta} \otimes t^{m+n}, \quad \forall \alpha, \beta, \alpha + \beta \in \dot{\Delta}, \\ [e_\alpha \otimes t^m, e_\beta \otimes t^n] &= 0, \quad \forall \alpha, \beta \in \dot{\Delta}, \alpha + \beta \notin \dot{\Delta} \cup \{0\}, \end{aligned}$$

where  $\gamma$  is the canonical linear space isomorphism from  $\dot{H}$  to  $\dot{H}^*$ .

### 3. VERTEX CONSTRUCTION OF LIE ALGEBRA OF TYPE $C_l^{(1)}$

**3.1.** Let  $H(m)$  ( $m \in \mathbb{Z}$ ) be an isomorphic copy of  $\dot{H}$ . Set  $\dot{H}_S := \text{Span}_{\mathbb{C}}\{\alpha_i \mid 1 \leq i \leq l-1\}$  and  $H_S(n - \frac{1}{2})$  ( $n \in \mathbb{Z}$ ) is an isomorphic copy of  $\dot{H}_S$ .

Define a Lie algebra

$$\hat{H} = \bigoplus_{m \in \mathbb{Z}} H(m) \oplus \bigoplus_{n \in \mathbb{Z}} H_S(n - \frac{1}{2}) \oplus \mathbb{C}c,$$

with Lie bracket

$$\begin{aligned} [\tilde{H}, c] &= 0, \\ [a(m), b(n)] &= m(a|b)\delta_{m,-n}c. \end{aligned}$$

Let

$$\hat{H}^- = \bigoplus_{m \in \mathbb{Z}_-} H(m) \oplus \bigoplus_{n \in \mathbb{Z}_-} H_S(n + \frac{1}{2}),$$

and let  $S(\hat{H}^-)$  be the symmetric algebra generated by  $\hat{H}^-$ . Then  $S(\hat{H}^-)$  is an  $\hat{H}$ -module with the action

$$c \cdot v = v, \quad a(m) \cdot v = a(m)v, \quad \forall m < 0,$$

and

$$a(m) \cdot b(n) = m(a, b)\delta_{m+n, 0}, \quad \forall m \geq 0, n < 0,$$

where  $a, b \in H$ ,  $m, n \in \frac{1}{2}\mathbb{Z}$ .

**3.2.** We form a group algebra  $\mathbb{C}[P]$  with base elements  $e^h$  ( $h \in P$ ), and the multiplication

$$e^{h_1}e^{h_2} = e^{h_1+h_2}, \quad \forall h_1, h_2 \in P.$$

Set

$$V(P) := S(\hat{H}^-) \otimes \mathbb{C}[P]$$

and extend the action of  $\hat{H}$  to space  $V(P)$  by

$$a(m) \cdot (v \otimes e^r) = (a(m) \cdot v) \otimes e^r, \quad \forall m \in \frac{1}{2}\mathbb{Z}^*;$$

and define

$$a(0) \cdot (v \otimes e^r) = (a|r)v \otimes e^r,$$

which makes  $V(P)$  into a  $\hat{H}$ -module.

**3.3.** For  $r \in P$ ,  $\alpha \in Q$ , define  $\mathbb{C}$ -linear operators as

$$\begin{aligned} e^\alpha \cdot (v \otimes e^r) &= v \otimes e^{\alpha+r}, \\ z^\alpha \cdot (v \otimes e^r) &= z^{(\alpha|r)} v \otimes e^r, \\ \epsilon_\alpha \cdot (v \otimes e^r) &= (-1)^{(s(\alpha)|r)} \epsilon_0(\alpha, r) v \otimes e^r, \\ a(z) &= \sum_{j \in \mathbb{Z}} a(j) z^{-2j}, \\ E^\pm(\alpha, z) \cdot (v \otimes e^r) &= \left( \exp\left(\mp \sum_{n=1}^{\infty} \frac{1}{n} z^{\mp 2n} \alpha(\pm n)\right) \cdot v \right) \otimes e^r, \\ F^\pm(\alpha, z) \cdot (v \otimes e^r) &= \left( \exp\left(\mp \sum_{n=0}^{\infty} \frac{2}{2n+1} z^{\mp(2n+1)} \alpha\left(\pm \frac{2n+1}{2}\right)\right) \cdot v \right) \otimes e^r. \end{aligned}$$

Then  $a(z), E^\pm(\alpha, z), F^\pm(\alpha, z) \in (\text{End}V(P))[[z, z^{-1}]]$ .

As usual, we shall adopt the notation of normal ordering product

$$: a(i)b(j) := \begin{cases} a(i)b(j), & \text{if } i \leq j, \\ b(j)a(i), & \text{if } j < i, \end{cases}$$

where  $a, b \in L$  and  $i, j \in \frac{1}{2}\mathbb{Z}$ .

**3.4.** Let  $\tilde{V}(P)$  be the formal completion of  $V(P) = S(\hat{H}^-) \otimes \mathbb{C}[P]$ . We give some vertex operators on  $\tilde{V}(P)$ :

(1) For  $\alpha \in \dot{\Delta} \cup \{0\}$ , set

$$Y(\alpha, z) = E^-(\alpha, z)E^+(\alpha, z)F^-(p(\alpha), z)F^+(p(\alpha), z),$$

$$Z^\epsilon(\alpha, z) = z^{(\alpha|\alpha)} e^\alpha z^{2\alpha} \epsilon_\alpha,$$

$$X^\epsilon(\alpha, z) := Y(\alpha, z) \otimes Z^\epsilon(\alpha, z).$$

(2) For  $\alpha, \beta \in \dot{\Delta}$ , define

$$X^\epsilon(\alpha, \beta, z, w) = : Y(\alpha, z)Y(\beta, w) : \otimes Z^\epsilon(\alpha + \beta, w).$$

**3.5.** The Laurent series of operators  $X^\epsilon(\alpha, z)$  is denoted by

$$X^\epsilon(\alpha, z) = \sum_{k=-\infty}^{\infty} X_{\frac{k}{2}}^\epsilon(\alpha) z^{-k}.$$

Then  $\forall k \in \mathbb{Z}$ ,  $X_{\frac{k}{2}}^\epsilon(\alpha)$  is an operator on  $V(P)$ . Note that  $X_n^\epsilon(\alpha)$  acts as an operator on  $V(P)$  in the following way:

$$X_n^\epsilon(\alpha) \cdot (v \otimes e^r) = \epsilon(\alpha, r) Y_{n+\frac{1}{2}(\alpha|\alpha)+(\alpha|r)}(\alpha)(v) \otimes e^{\alpha+r}, \quad \forall v \otimes e^r \in V(P).$$

**3.6.** For  $v = a_1(-n_1)a_2(-n_2)\cdots a_p(-n_p) \otimes e^r \in V(L)$ , define the degree action of  $d$  on  $V(P)$  by

$$d \cdot (v \otimes e^r) = \left( \deg(v) - \frac{1}{2}(r|r) \right) v \otimes e^r,$$

where  $\deg(v) = -\sum_{i=1}^p n_i$ .

The number  $\deg(v) - \frac{1}{2}(r|r)$  is called the degree of  $v \otimes e^r$  and denoted by  $\deg(v \otimes e^r)$ .

**3.7.**

**Proposition 3.1.** *The affine Lie algebra  $\tilde{\mathcal{G}}$  of type  $C_l^{(1)}$  is homomorphic to the Lie algebra  $J$  generated by operators  $\alpha^\vee(n), X_n^\epsilon(\alpha), c, d$  ( $\alpha \in \dot{\Delta}, n \in \mathbb{Z}$ ) on  $V(P) = S(\hat{H}^-) \otimes \mathbb{C}[P]$ , i.e., there exists a unique Lie algebra homomorphism  $\pi$  from  $\tilde{\mathcal{G}}$  to the Lie subalgebra  $J$  of  $\text{End}(V(P))$  such that*

$$\begin{aligned} \pi(\gamma^{-1}(\alpha_i) \otimes t^n) &= \frac{2}{(\alpha_i|\alpha_i)} \alpha_i(n), \\ \pi(e_\alpha \otimes t^n) &= X_n^\epsilon(\alpha), \\ \pi(c) &= \text{id}, \\ \pi(d) &= d, \end{aligned}$$

that is,  $V(P)$  is a  $\tilde{\mathcal{G}}$ -module.

#### 4. SOME COMPUTATIONS NEEDED

**Lemma 4.1.** *For any  $l$ , if  $\Lambda_s$  is the basic weight of  $C_l^{(1)}$ , then we have*

$$(4.1) \quad \dim_q(L(\Lambda_s)) = \dim_q(L(\Lambda_{l-s})),$$

$$(4.2) \quad \dim_q(L(\Lambda_s)) = \prod_{n=1}^{\infty} \frac{(1 - q^{2(l+2)n})(1 - q^{2(l+2)n-2-2s})(1 - q^{2(l+2)n-2l-2+2s})}{(1 - q^n)}.$$

For the definition of  $\dim_q$ , one can refer [7] (see p. 183, Proposition 10.10).

Define  $q$ -series

$$(4.3) \quad \kappa_q(l, r) = \sum_{n \in \mathbb{Z}} q^{ln^2 - rn},$$

for  $0 < r \leq l$ . If  $r = l$ , then

$$(4.4) \quad \kappa_q(l, l) = 2 \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^2}{(1 - q^{2n})},$$

by Gauss identity

$$(4.5) \quad \sum_{n \in \mathbb{Z}} q^{2n^2 - n} = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^n)}.$$

Suppose that  $V = S(\alpha(-1), \alpha(-2), \dots) \otimes \mathbb{C}[\mathbb{Z}\alpha]$  with  $(\alpha|\alpha) = 2$ , then  $V$  is an irreducible  $A_1^{(1)}$ -module isomorphic to  $L(\Lambda_0)$  (one can see [FLM] for details). The degree of  $v = \alpha(-n_1) \cdots \alpha(-n_k) \otimes e^{n\alpha} \in V$  is defined as  $-n_1 - \cdots - n_k - n^2$  and weight of  $v$  is  $-(n_1 + \cdots + n_k + n^2)\delta + n\alpha$ . Hence

$$(4.6) \quad \text{ch} V = e^{\Lambda_0} \frac{1}{\prod_{n=1}^{\infty} (1 - e^{-n\delta})} \sum_{n \in \mathbb{Z}} e^{-n^2\delta + n\alpha}.$$

Moreover,

$$(4.7) \quad \text{ch} L(\Lambda_0) = e^{\Lambda_0} \frac{\sum_{n \in \mathbb{Z}} e^{-(3n^2+n)\delta + 3n\alpha} - \sum_{n \in \mathbb{Z}} e^{-(3n^2+n)\delta - (3n+1)\alpha}}{\prod_{n=1}^{\infty} (1 - e^{-n\delta})(1 - e^{-n\delta + \alpha})(1 - e^{-(n-1)\delta - \alpha})}.$$

If  $e^{-\delta}, e^{-\alpha}$  are specialized as  $q^l, q^r$ , respectively, then  $V \cong L(\Lambda_0)$  implies:

**Lemma 4.2.** *If  $0 < r < l$ , then*

$$(4.8) \quad \kappa_q(l, r) = \prod_{n=1}^{\infty} \frac{(1 - q^{2ln})(1 - q^{4ln-2(l-r)})(1 - q^{4l(n-1)+2(l-r)})}{(1 - q^{2ln-l-r})(1 - q^{2l(n-1)+l+r})}.$$

*Proof.* This lemma can easily be proved using the quintuple product identity (see [3]).  $\square$

#### 5. THE MODULE STRUCTURE

**5.1.** Let  $\alpha_0 \in H^*$  such that  $\{\alpha_0, \alpha_1, \dots, \alpha_l\}$  is the simple root system of affine Lie algebra  $\tilde{\mathcal{G}}$  and  $\alpha_0(\alpha_0^\vee) = 2$ ,  $\alpha_0(\alpha_1^\vee) = -2$ ,  $\alpha_0(d) = 1$  and  $\alpha_0(\alpha_j^\vee) = \alpha_0(c) = 0$  ( $2 \leq j \leq l$ ). Then  $\delta = \alpha_0 + 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l$  is the primitive imaginary root of  $\tilde{\mathcal{G}}$ . Let  $\Lambda_i \in H^*$  be such that

$$\Lambda_i(\alpha_j^\vee) = \delta_{ij}, \quad \Lambda_i(d) = 0 \quad (0 \leq j \leq l).$$



**Lemma 5.1.** *With respect to the Cartan subalgebra  $H$  of  $\tilde{\mathcal{G}}$ ,  $V(P)$  has the weight space decomposition*

$$V(P) = \sum_{\lambda \in \text{weight}(V(P))} V(P)_\lambda,$$

and the weight space  $V(P)_\lambda$  has a basis  $v \otimes e^r$ , where  $r \in P$ ,  $v \in S(\dot{\mathcal{H}}^-)$ , and

$$\lambda = \Lambda_0 + \left( \deg(v) - \frac{1}{2}(r | r) \right) \delta + r,$$

so  $\deg(v)$  and  $r$  are uniquely determined by  $\lambda$ .

**5.2.** The following describes the possible distribution of the maximal weights of  $\tilde{\mathcal{G}}$ -module  $V(\dot{Q})$ .

**Lemma 5.2.** *For any  $\lambda \in P(V(\dot{Q}))$ , we have*

$$\lambda \leq \Lambda_j - \frac{j}{4}\delta,$$

for some  $j \in \mathbb{Z}$ , where  $0 \leq j \leq l$ .

*Proof.* By Lemma 5.1,  $\lambda = \Lambda_0 - (k + \frac{1}{2}(r | r))\delta + r$ , where  $r = \sum_{i=1}^{l-1} k_i \alpha_i + \frac{k_l}{2} \alpha_l \in P$  and  $k \in \frac{1}{2} \mathbb{N}$ . At first, we have

$$\begin{aligned} \frac{1}{2}(r | r)\delta - r &= \frac{1}{4}(k_1^2 + (k_2 - k_1)^2 + \cdots + (k_{l-1} - k_{l-2})^2 + (2k_l - k_{l-1})^2)\delta - \sum_{i=1}^{l-1} k_i \alpha_i - \frac{k_l}{2} \alpha_l \\ &= \frac{1}{4}[(k_1^2 \delta - 2k_1(2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l)) \\ &\quad + ((k_2 - k_1)^2 \delta - 2(k_2 - k_1)(2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l)) \\ &\quad \cdots \\ &\quad + ((k_{l-1} - k_{l-2})^2 \delta - 2(k_{l-1} - k_{l-2})(2\alpha_{l-1} + \alpha_l)) \\ &\quad + (k_l - k_{l-1})^2 \delta - 2(k_l - k_{l-1})\alpha_l]. \end{aligned}$$

Suppose that

$$\alpha = 2\alpha_i + \cdots + 2\alpha_{l-1} + \alpha_l < \delta.$$

If  $n < 0$ , then  $n^2\delta - 2n\alpha > 0$ . If  $n > 1$ , then

$$(n^2 - 1)\delta - 2(n - 1)\alpha = (n - 1)((n + 1)\delta - 2\alpha) > 0.$$

Hence we have

$$n^2\delta - 2n\alpha \geq 0$$

or

$$n^2\delta - 2n\alpha \geq \delta - 2\alpha.$$

So

$$\begin{aligned} \frac{1}{2}(r | r)\delta - r &\geq \frac{s}{4}\delta - \frac{1}{2}[(2\alpha_{p_1} + \cdots + 2\alpha_{l-1} + \alpha_l) + \cdots + (2\alpha_{p_s} + \cdots + 2\alpha_{l-1} + \alpha_l)] \\ &\geq \frac{s}{4}\delta - \frac{1}{2}[(2\alpha_1 + \cdots + 2\alpha_{l-1} + \alpha_l) + \cdots + (2\alpha_s + \cdots + 2\alpha_{l-1} + \alpha_l)] \\ &= \frac{1}{2}(\gamma_s | \gamma_s)\delta - \gamma_s, \end{aligned}$$

for some  $s$ , where

$$\gamma_s = \alpha_1 + 2\alpha_2 + \cdots + (s - 1)\alpha_{s-1} + s(\alpha_s + \cdots + \alpha_{l-1}) + \frac{s}{2}\alpha_l \in P,$$

and it clear that  $\Lambda_s = \Lambda_0 + \gamma_s$ ,  $(\gamma_s | \gamma_s) = \frac{s}{2}$ . Then we have

$$\begin{aligned} \lambda &= \Lambda_0 - \left( k + \frac{1}{2}(r | r) \right) \delta + r \\ &\leq \Lambda_0 - \frac{1}{2}(r | r) \delta + r \\ &\leq \Lambda_0 - \frac{1}{2}(\gamma_s | \gamma_s) \delta + \gamma_s \\ &= \Lambda_s - \frac{s}{4} \delta \end{aligned}$$

for some  $s$  ( $0 \leq s \leq l$ ). □

**Remark 5.3.** By the result above, we know that any highest weight of  $V(P)$  belongs to the set

$$\bigcup_{s=0}^l \left\{ \Lambda_s - \frac{s}{4} \delta - \frac{p}{2} \delta \mid p \geq 0, s = 0, 1, \dots, l \right\}.$$

More precisely, any highest weight vector has the form  $v \otimes e^{\gamma_s}$  for some  $s$ .

**Theorem 5.4.**  $V(P)$  has the decomposition

$$V(P) = \bigoplus_{s=0}^l V(P)^{[s]},$$

where  $V(P)^{[s]}$  is the sum of those irreducible submodules whose highest weights  $\lambda \leq \Lambda_s - \frac{s}{4} \delta$ .

## 6. HIGHEST WEIGHT VECTORS

**6.1.** Define operators

$$(6.1) \quad S(\alpha, z) = \exp \left( \sum_{n>0} \frac{\alpha(-n + \frac{1}{2})}{n - \frac{1}{2}} z^{2n-1} \right) \exp \left( - \sum_{n>0} \frac{\alpha(n - \frac{1}{2})}{n - \frac{1}{2}} z^{-2n+1} \right),$$

with series expansion

$$(6.2) \quad S(\alpha, z) = \sum_{n \in \frac{1}{2}\mathbb{Z}} S_n(\alpha) z^{-2n}.$$

**Lemma 6.1.** For  $i = 1, \dots, l-1$ , we have

$$\{S_n(\alpha_i), S_m(\alpha_i)\} = S_n(\alpha_i)S_m(\alpha_i) + S_m(\alpha_i)S_n(\alpha_i) = -2\delta_{m+n,0}$$

and

$$S_n(\alpha) = (-1)^{2n} S_n(-\alpha), \quad n \in \frac{1}{2}\mathbb{Z}.$$

**6.2.** Define  $\beta_i = \alpha_i$  for  $i = 1, \dots, l-1$  and

$$(6.3) \quad \beta_l = - \sum_{i=1}^{l-1} \frac{i}{l} \alpha_i,$$

also let

$$(6.4) \quad y_i = \sum_{j=i}^l 2\beta_j, \quad i = 1, \dots, l.$$

Define

$$(6.5) \quad Z^{[s]}(z) = \sum_{i \in \mathbb{Z}} Z_{\frac{i}{2}}^{[s]} z^{-i} = \sum_{j=1}^s S(y_j, z) - \sum_{j=s+1}^l S(y_j, z),$$

for even  $s$ . Particularly,  $Z^{[l]}(z) = -Z^{[0]}(z)$ .

**Remark 6.2.** *The operators  $Z^{[s]}$  are the same as (or isomorphic to) those defined by Lepowsky and Wilson in [8], [9], where they are generating operators of vacuum spaces of standard  $A_1^{(1)}$ -modules of level  $l$ . For more details, one can refer to those two papers.*

**Lemma 6.3.** *For any  $n \in \frac{1}{2}\mathbb{Z}$ , if  $v \otimes e^{\gamma^s}$  is a highest weight vector and  $Z_n^{[s]}v \otimes e^{\gamma^s}$  is not zero, then  $Z_n^{[s]}v \otimes e^{\gamma^s}$  is also a highest weight vector.*

*Proof.* At first, we give the proof for  $s = 0$ . For  $i < l$ , we have

$$S_n(y_i) + S_n(y_{i+1}) = \sum_{j \in \mathbb{Z}} S_j(\beta_i) S_{n-j}(\beta_i + 2\beta_{i+1} + \cdots + 2\beta_l),$$

and  $(y_j | \alpha_i) = 0$ ,  $j \neq i, i+1$ . Hence

$$\begin{aligned} -X_0^\epsilon(\alpha_i) Z_n^{[0]}(v \otimes 1) &= X_0^\epsilon(\alpha_i) \left\{ \sum_{r \neq i, i+1} S_n(y_r) + \sum_{j \in \mathbb{Z}} S_j(\beta_i) S_{n-j}(\beta_i + 2\beta_{i+1} + \cdots + 2\beta_l) \right\} v \otimes 1 \\ &= Y_{\frac{1}{2}}(\alpha_i) \left\{ \sum_{r \neq i, i+1} S_n(y_r) + \sum_{j \in \mathbb{Z}} S_j(\beta_i) S_{n-j}(\beta_i + 2\beta_{i+1} + \cdots + 2\beta_l) \right\} v \otimes e^{\alpha_i} \\ &= Y_{\frac{1}{2}}(\alpha_i) \sum_{r \neq i, i+1} S_n(y_r) v \otimes e^{\alpha_i} \\ &\quad + Y_{\frac{1}{2}}(\alpha_i) \sum_{j \in \mathbb{Z}} S_j(\alpha_i) S_{n-j}(\beta_i + 2\beta_{i+1} + \cdots + 2\beta_l) v \otimes e^{\alpha_i} \\ &= \sum_{r \neq i, i+1} S_n(y_r) X_0^\epsilon(\alpha_i) v \otimes 1 \\ &\quad + \sum_{k \in \mathbb{Z}} E_k(\alpha_i) S_{\frac{1}{2}-k}(\alpha_i) \sum_{j \in \mathbb{Z}} S_j(\alpha_i) S_{n-j}(\beta_i + 2\beta_{i+1} + \cdots + 2\beta_l) v \otimes e^{\alpha_i} \\ &= - \sum_{j \in \mathbb{Z}} S_j(\alpha_i) S_{n-j}(\beta_i + 2\beta_{i+1} + \cdots + 2\beta_l) X_0^\epsilon(\alpha_i) v \otimes 1 = 0. \end{aligned}$$

Moreover, operators  $X_0^\epsilon(\alpha_l)$  and  $X_1^\epsilon(-(2\alpha_1 + \cdots + 2\alpha_{l-1} + \alpha_l))$  commute with  $Z_n^{[0]}$ , so  $Z_n^{[0]}v \otimes 1$  is still a highest weight vector.

The proof for  $s = l$  is the same as above.

For  $Z_n^{[s]}$  with  $0 < s < l$ ,

$$S_n(y_i) - S_n(y_{i+1}) = \sum_{j \in \mathbb{Z} + \frac{1}{2}} S_j(\beta_i) S_{n-j}(\beta_i + 2\beta_{i+1} + \cdots + 2\beta_l),$$

then

$$\begin{aligned}
& X_0^\epsilon(\alpha_s) Z_n^{[s]}(v \otimes e^{\gamma_s}) \\
&= X_0^\epsilon(\alpha_s) \left\{ \left( \sum_{r < s} - \sum_{r > s+1} \right) S_n(y_r) + \sum_{j \in \mathbb{Z} + \frac{1}{2}} S_j(\beta_s) S_{n-j}(\beta_s + 2\beta_{s+1} + \cdots + 2\beta_l) \right\} v \otimes e^{\gamma_s} \\
&= Y_1(\alpha_s) \left\{ \left( \sum_{r < s} - \sum_{r > s+1} \right) S_n(y_r) + \sum_{j \in \mathbb{Z}} S_j(\beta_s) S_{n-j}(\beta_s + 2\beta_{s+1} + \cdots + 2\beta_l) \right\} v \otimes e^{\gamma_s + \alpha_i} \\
&= Y_1(\alpha_i) \left( \sum_{r < s} - \sum_{r > s+1} \right) S_n(y_r) v \otimes e^{\gamma_s + \alpha_i} \\
&\quad + Y_1(\alpha_i) \sum_{j \in \mathbb{Z}} S_j(\alpha_s) S_{n-j}(\beta_s + 2\beta_{s+1} + \cdots + 2\beta_l) v \otimes e^{\gamma_s + \alpha_s} \\
&= \sum_{r \neq i, i+1} S_n(y_r) X_0^\epsilon(\alpha_s) v \otimes e^{\gamma_s} \\
&\quad + \sum_{k \in \mathbb{Z}} E_k(\alpha_i) S_{1-k}(\alpha_i) \sum_{j \in \mathbb{Z} + \frac{1}{2}} S_j(\alpha_s) S_{n-j}(\beta_s + 2\beta_{s+1} + \cdots + 2\beta_l) v \otimes e^{\gamma_s + \alpha_s} \\
&= - \sum_{j \in \mathbb{Z}} S_j(\alpha_s) S_{n-j}(\beta_s + 2\beta_{s+1} + \cdots + 2\beta_l) X_0^\epsilon(\alpha_s) v \otimes e^{\gamma_s} = 0.
\end{aligned}$$

For other  $X^\epsilon(\alpha_i)$  and  $X_0^\epsilon(\alpha_l)$ ,  $X_1^\epsilon(-(2\alpha_1 + \cdots + 2\alpha_{l-1} + \alpha_l))$ , the proof is similar to the first case. Then  $Z_n^{[s]}v \otimes e^{\gamma_s}$  is also a highest weight vector.  $\square$

For  $\lambda = \Lambda_0 - \sum_{i=0}^l k_i \alpha_i$ , define

$$\deg \lambda = \sum_{i=0}^l k_i,$$

and

$$V(P)_i = \sum_{\lambda: \deg \lambda = i} V(P)_\lambda,$$

then

$$V(P) = \sum V(P)_i.$$

The  $q$ -character  $\text{ch}_q$  is a map from  $V(P)$  to  $\mathbb{Z}[q^{\pm 1}]$  (to  $\mathbb{Z}[q^{\pm \frac{1}{2}}]$  if  $l$  is even) defined by

$$(6.6) \quad \text{ch}_q V(P) = \sum \dim V(P)_i q^i.$$

Define the highest weight vector space of  $V(P)^{[s]}$  as  $\Omega_s \otimes e^{\gamma_s}$ . Then we have

**Theorem 6.4.**  $\Omega_s$  is generated by operators  $Z_i^{[s]} (i \in \frac{1}{2}\mathbb{Z}_-)$ . Moreover,

$$(6.7) \quad \text{ch}_q \Omega_s = \prod_{n=1}^{\infty} \frac{(1 - q^{l(l+2)n})(1 - q^{l(l+2)n-s-1})(1 - q^{l[(l+2)n-l+s-1]})}{(1 - q^{ln})}.$$

## 7. PROOF OF THEOREM 6.4

Let

$$(7.1) \quad \hat{H}^- = \bigoplus_{n \in \mathbb{Z}_-} H_S(n + \frac{1}{2}).$$

Theorem 6.4 will be proved by the following lemmas.

**Lemma 7.1.**  $S(\widehat{H_S^-}) \otimes 1$  can be generated by operators  $Z_n^{[s]} (n \in \frac{1}{2}\mathbb{Z}, s = 0, \dots, l)$  on  $1 \otimes 1$ .

*Proof.* At first, by the definition of operators  $Z^{[s]}(z)$ ,

$$\begin{aligned} Z_n^{[1]} - Z_n^{[0]} &= 2S(y_1), \\ Z_n^{[2]} - Z_n^{[1]} &= 2S(y_2), \\ &\dots \\ Z_n^{[l-1]} - Z_n^{[l-2]} &= 2S(y_{l-1}), \end{aligned}$$

moreover, for  $0 < s < l$  and  $m \in \mathbb{Z}$ ,  $y_s(m + \frac{1}{2})$  can be generated by operators  $S_n(y_s)(n \in \frac{1}{2}\mathbb{Z})$ . So  $S(H_S^-) \otimes 1$  can be generated by the  $Z_n^{[s]}$ 's.  $\square$

**Lemma 7.2.** Suppose that  $v \in S(\widehat{H_S^-})$ , then  $v \otimes e^{\gamma_s}$  is a highest weight vector if and only if for all positive integers  $m$ ,

$$S_{m-\frac{1}{2}}(\alpha_i)v \otimes 1 = 0 \quad (0 < i < l, i \neq s), \quad S_m(\alpha_s)v \otimes 1 = 0 \quad (\text{when } (\alpha_s | \alpha_s) = 1).$$

*Proof.* As we know that  $v \otimes e^{\gamma_s}$  is a highest weight vector if and only if

$$X_0^\epsilon(\alpha_i)v \otimes e^{\gamma_s} = X_1^\epsilon(-2\alpha_1 - \dots - 2\alpha_{l-1} - \alpha_l)v \otimes e^{\gamma_s} = 0, \quad i = 1, \dots, l.$$

For any  $v \in S(\widehat{H_S^-})$ , it always holds that

$$X_0^\epsilon(\alpha_l)v \otimes e^{\gamma_s} = X_1^\epsilon(-2\alpha_1 - \dots - 2\alpha_{l-1} - \alpha_l)v \otimes e^{\gamma_s} = 0.$$

Let

$$E(\alpha, z) = E^-(\alpha, z)E^+(\alpha, z) = \sum_{j \in \mathbb{Z}} E_j(\alpha)z^{-j},$$

then for  $0 < i < l$ ,

$$X_0^\epsilon(\alpha_i)v \otimes e^{\gamma_s} = \epsilon_{\alpha_i} Y_{\frac{1}{2}}(\alpha_i)v \otimes e^{\gamma_s + \alpha_i} = \epsilon_{\alpha_i} \sum_{j \in \mathbb{Z}} E_j(\alpha_i)S_{\frac{1}{2}-j}(\alpha_i)v \otimes e^{\gamma_s + \alpha_i}$$

for  $i \neq s$  and

$$X_0^\epsilon(\alpha_i)v \otimes e^{\gamma_s} = \epsilon_{\alpha_i} Y_1(\alpha_i)v \otimes e^{\gamma_s + \alpha_i} = \epsilon_{\alpha_i} \sum_{j \in \mathbb{Z}} E_j(\alpha_i)S_{1-j}(\alpha_i)v \otimes e^{\gamma_s + \alpha_i}$$

for  $i = s$ . Thus this lemma holds.  $\square$

**Lemma 7.3.** If  $v \in S(\widehat{H_S^-})$  and for all positive integer  $m$ ,

$$S_{m-\frac{1}{2}}(\alpha_1)v \otimes 1 = 0,$$

then  $v$  belongs to the subspace  $W_1$  generated by  $Z_{\frac{n}{2}}^{[0]}, Z_{\frac{n}{2}}^{[2]}, \dots, Z_{\frac{n}{2}}^{[l-1]}, Z_{\frac{n}{2}}^{[l]} = -Z_{\frac{n}{2}}^{[0]} (n \in \mathbb{Z})$ .

*Proof.* Notice that

$$\begin{aligned} Z_{\frac{n}{2}}^{[0]} &= - \sum_{r \neq 1, 2} S_{\frac{n}{2}}(y_r) - \sum_{j \in \mathbb{Z}} S_j(\beta_1)S_{\frac{n}{2}-j}(\beta_1 + 2\beta_2 + \dots + 2\beta_l) \\ &= - \sum_{j \in \mathbb{Z}} S_j(\alpha_1)S_{\frac{n}{2}-j}(\beta_1 + 2\beta_2 + \dots + 2\beta_l) + \text{terms commuting with } S(\alpha_1). \end{aligned}$$

$$Z_{\frac{n}{2}}^{[1]} = \sum_{j \in \mathbb{Z}} S_{j+\frac{1}{2}}(\alpha_1)S_{\frac{n}{2}-j-\frac{1}{2}}(\beta_1 + 2\beta_2 + \dots + 2\beta_l) + \text{terms commuting with } S(\alpha_1).$$

and

$$Z_{\frac{n}{2}}^{[s]} = \sum_{j \in \mathbb{Z}} S_j(\alpha_1) S_{\frac{n}{2}-j}(\beta_1 + 2\beta_2 + \cdots + 2\beta_l) + \text{terms commuting with } S(\alpha_1),$$

for  $s \geq 2$ . Since  $(\alpha_1 | \beta_1 + 2\beta_2 + \cdots + 2\beta_l) = 0$ , a homogeneous non-zero vector

$$v = v \otimes 1 = \sum a_{j_1, \dots, j_k} Z_{j_1}^{[s_1]} \cdots Z_{j_k}^{[s_k]} \otimes 1$$

can be written as

$$\sum b_{i_1, \dots, i_r} S_{i_1}(\alpha_1) \cdots S_{i_r}(\alpha_1) \otimes 1, (i_1 < \cdots < i_r \leq 0)$$

where  $b_{i_1, \dots, i_r}$  is a non-zero polynomial commuting with  $S(\alpha_1)$ . Then  $v \in W_1$  if and only if  $i_1, \dots, i_r \in \mathbb{Z}$  for any  $b_{i_1, \dots, i_r}$ . It is easy to show that if  $b_{i_1, \dots, i_r} \otimes 1 \neq 0$ , then

$$S_{-j_1}(\alpha_1) \cdots S_{-j_r}(\alpha_1) v = \text{a scalar of } b_{j_1, \dots, j_r} \otimes 1 \neq 0.$$

Condition  $S_{m-\frac{1}{2}}(\alpha_1) v \otimes 1 = 0$  implies all  $i_1, \dots, i_r \in \mathbb{Z}$ , so  $v \in W_1$ .  $\square$

A similar argument shows the following two lemmas.

**Lemma 7.4.** *If  $v \in S(\widehat{H_S}^-)$  and for all positive integer  $m$ ,*

$$S_{m-\frac{1}{2}}(\alpha_1) v \otimes 1 = 0, S_{m-\frac{1}{2}}(\alpha_2) v \otimes 1 = 0,$$

*then  $v$  belongs to the subspace generated by  $Z_{\frac{n}{2}}^{[0]}, Z_{\frac{n}{2}}^{[3]}, \dots, Z_{\frac{n}{2}}^{[l-1]}, Z_{\frac{n}{2}}^{[l]} = -Z_{\frac{n}{2}}^{[0]}$ .*

**Lemma 7.5.** *If  $v \in S(\widehat{H_S}^-)$  and for all positive integer  $m$  and  $1 < i < l$ ,*

$$S_{m-\frac{1}{2}}(\alpha_i) v \otimes 1 = 0,$$

*then  $v$  belongs to the subspace generated by  $Z_n^{[0]}$ .*

Similarly to the proof for  $s = 0$  above, for general  $s$ , we have

**Lemma 7.6.** *Suppose that  $v \in S(\widehat{H_S}^-)$  and  $0 < s < l$ . If*

$$S_{m-\frac{1}{2}}(\alpha_i) v \otimes 1 = 0 (0 < i < l, i \neq s), \quad S_m(\alpha_s) v \otimes 1 = 0 (\text{when } (\alpha_s | \alpha_s) = 1),$$

*for all positive integer  $m$ , then  $v$  belongs to the subspace generated by  $Z_n^{[s]}$ .*

**Lemma 7.7.** *For any  $0 \leq s \leq l$ , the element  $1 \otimes e^{\gamma_s}$  is a highest weight vector.*

**Lemma 7.8.** *For odd  $l \geq 3$ ,  $\Omega_s$  has basis*

$$\left\{ Z_{n_1}^{[s]} \cdots Z_{n_k}^{[s]} \otimes 1 \mid n_p \in \frac{1}{2}\mathbb{Z}_-, n_p \leq n_{p+1}, n_p \leq n_{p+r} - 1, n_{k-\sigma(s)} \leq -1 \right\}.$$

*For even  $l \geq 2$ ,  $\Omega_s$  has basis*

$$\left\{ Z_{n_1}^{[s]} \cdots Z_{n_k}^{[s]} \otimes 1 \mid n_p \in \frac{1}{2}\mathbb{Z}_-, n_p - n_{p+r} < -1 \Rightarrow \sum_{i=0}^r n_{p+i} \in \mathbb{Z}, n_p \leq n_{p+r} - 1, n_{k-\sigma(i)} \leq -1, \right\},$$

*here  $r = \frac{l-1}{2}$  if  $l$  odd and  $r = \frac{l}{2}$  if  $l$  even.  $\sigma(s) = s$  for  $s \leq r$ , otherwise,  $\sigma(s) = r + 1 - s$ .*

**Lemma 7.9.** *For any  $0 \leq s \leq l$ ,*

$$\text{ch}_q \Omega_s = \prod_{n=1}^{\infty} \frac{(1 - q^{l(l+2)n})(1 - q^{l[(l+2)n-s-1]})(1 - q^{l[(l+2)n-l+s-1]})}{(1 - q^{ln})}.$$

For Lemmas 7.8 and 7.9, one can refer ([8], Theorem 10.4), ([9], Section 14) and ([2], Section 3).

Lemmas 7.1—7.9 prove Theorem 6.4.

## 8. PRODUCT-SUM IDENTITIES

Since

$$(8.1) \quad V(P) = \sum_{s=0}^l \Omega_s \otimes L(\Lambda_s - \frac{s}{4}\delta),$$

we have the specialized character

$$(8.2) \quad \text{ch}_q V(P) = \sum_{s=0}^l \text{ch}_q \Omega_s \text{ch}_q L(\Lambda_s - \frac{s}{4}\delta),$$

the L.H.S. is

$$(8.3) \quad \frac{\sum_{n_1, \dots, n_l \in \mathbb{Z}} q^{\frac{1}{2}(ln_1^2 - n_1 + ln_2^2 - 3n_2 \dots + ln_l^2 - (2l-1)n_l)}}{\prod_{n=1}^{\infty} (1 - q^{ln})^{l-1} (1 - q^{2ln})}$$

which equals

$$(8.4) \quad \frac{q^{-\frac{l^2}{8}} [\kappa_{q^{\frac{1}{2}}}(l, 1) \kappa_{q^{\frac{1}{2}}}(l, 3), \dots, \kappa_{q^{\frac{1}{2}}}(l, l-1)]^2}{\prod_{n=1}^{\infty} (1 - q^{ln})^{l-1} (1 - q^{2ln})} = q^{-\frac{l^2}{8}} \prod_{n=1}^{\infty} \frac{(1 + q^{n-\frac{1}{2}})^2}{(1 - q^{ln})}$$

for even  $l$ , and equals

$$(8.5) \quad \frac{q^{-\frac{l^2-1}{8}} [\kappa_{q^{\frac{1}{2}}}(l, 1) \kappa_{q^{\frac{1}{2}}}(l, 3), \dots, \kappa_{q^{\frac{1}{2}}}(l, l-2)]^2 \kappa_{q^{\frac{1}{2}}}(l, l)}{\prod_{n=1}^{\infty} (1 - q^{ln})^{l-1} (1 - q^{2ln})} = 2q^{-\frac{l^2-1}{8}} \prod_{n=1}^{\infty} \frac{(1 - q^{2ln-l})}{(1 - q^{2n-1})^2}$$

for odd  $l$ . Where  $\kappa_q$  is defined by Eqs. (4.3) and (4.4).

The R.H.S. is

$$(8.6) \quad \sum_{s=0}^l \text{ch}_q \Omega_s \text{ch}_q L(\Lambda_s - \frac{s}{4}\delta) = \sum_{s=0}^l q^{\frac{(l-s)s}{2}} \text{ch}_q \Omega_s \dim_q L(\Lambda_s).$$

Then by the computation of  $\Omega_s$  and  $\dim_q L(\Lambda_s)$  before, the proof for our main theorems is finished.

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